

RESTRICTED BEURLING TRANSFORMS ON CAMPANATO SPACES

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ABSTRACT. Let $\Omega \subset \mathbb{C}$ be a bounded domain with $\mathcal{C}^{1,\omega}$ -smooth boundary, where ω is a Dini-smooth modulus of continuity. We prove that the restricted Beurling transform is bounded on the Campanato space $\text{BMO}_\omega(\Omega)$.

1. INTRODUCTION

The Beurling transform is the principal value convolution operator

$$Bf(z) = -\frac{1}{\pi} \text{p.v.} \int_{\mathbb{C}} f(z-w) \frac{1}{w^2} dw.$$

Given a bounded domain $\Omega \subset \mathbb{C}$, in the present paper we consider the corresponding modification of B . Namely, the restricted Beurling transform B_Ω is defined as

$$B_\Omega f(z) = B(\chi_\Omega f)(z), \quad z \in \Omega,$$

where χ_Ω is the characteristic function of Ω .

1.1. Motivation: applications to quasiregular mappings. The studies of B_Ω are motivated by applications in the theory of quasiregular mappings (see [1]). In the setting of the classical Lipschitz spaces $\Lambda^\alpha(\Omega)$, $0 < \alpha < 1$, the following result is known to be crucial; see [11].

Theorem 1 (see [11, Main Lemma]). *Let $\Omega \subset \mathbb{C}$ be a bounded domain with $\mathcal{C}^{1+\alpha}$ -smooth boundary, $0 < \alpha < 1$. Then the restricted Beurling transform B_Ω is bounded on $\Lambda^\alpha(\Omega)$.*

In particular, the above theorem is applied in [11] to study the principal solutions of the Beltrami equation

$$\bar{\partial}f = \mu \partial f, \quad \text{a.e. on } \Omega,$$

under assumption that the Beltrami coefficient μ , $\|\mu\|_{L^\infty(\Omega)} < 1$, is in $\Lambda^\alpha(\Omega)$, $0 < \alpha < 1$. See also [4], where the Lipschitz conditions are replaced by Sobolev or Besov conditions.

Motivated by the applications mentioned above, we are looking for extensions of Theorem 1 to larger classes of bounded domains $\Omega \subset \mathbb{C}$. A more precise problem is formulated as follows: Under weaker restrictions on $\partial\Omega$, find B_Ω -invariant spaces $X(\Omega)$, $X(\Omega) \subset L^\infty(\Omega)$. To solve the problem, we consider spaces defined by means of regular moduli of continuity.

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1.2. Moduli of continuity.

Definition 1. An increasing continuous function $\omega : [0, +\infty) \rightarrow [0, +\infty)$, $\omega(0) = 0$, is called a *modulus of continuity*.

The Lipschitz space $\Lambda^\omega(\Omega)$ consists of those functions $f : \Omega \rightarrow \mathbb{C}$ for which there exists a constant $C > 0$ such that

$$|f(z) - f(w)| \leq C\omega(|z - w|), \quad z, w \in \Omega.$$

For an interval $(a, b) \subset \mathbb{R}$, the space $\Lambda^\omega(a, b)$ is defined analogously. Also, we write $\varphi \in \mathcal{C}^{1,\omega}(a, b)$ if φ is differentiable on (a, b) and $\varphi' \in \Lambda^\omega(a, b)$.

A modulus of continuity ω is called *Dini-smooth* if the integral

$$\int_0^1 \frac{\omega(t)}{t} dt$$

converges.

Also, we use the following regularity condition: there exists $\varepsilon \in (0, 1)$ such that the quotient $\omega(t)/t^\varepsilon$ is *almost decreasing*, that is,

$$(1.1) \quad \frac{\omega(t)}{t^\varepsilon} \leq C \frac{\omega(s)}{s^\varepsilon}, \quad t > s > 0,$$

for a universal constant $C > 0$.

Definition 2. A modulus of continuity ω is called *regular* if ω is Dini-smooth and property (1.1) holds.

In what follows, we assume that ω is regular if not otherwise stated. For $\beta > 0$, the logarithmic function

$$\left(\log \frac{e}{t}\right)^{-1-\beta}, \quad 0 < t < 1,$$

may serve as a working example of a regular modulus of continuity restricted to $(0, 1)$.

1.3. Restricted Beurling transform on $\Lambda^\omega(\Omega)$. Given a modulus of continuity ω , we say that $\Omega \subset \mathbb{C}$ is a domain with $\mathcal{C}^{1,\omega}$ -smooth boundary if $\partial\Omega$ is a \mathcal{C}^1 curve whose unit normal vector is in Λ^ω as a function on the curve. An equivalent technical reformulation of this assumption is given in Section 2.2.

Proposition 2 ([17, Theorem 1]). *Let ω be a Dini-smooth modulus of continuity and let $\Omega \subset \mathbb{C}$ be a bounded domain with $\mathcal{C}^{1,\omega}$ -smooth boundary. Then B_Ω maps $\Lambda^\omega(\Omega)$ into $\Lambda^{\tilde{\omega}}(\Omega)$, where*

$$\tilde{\omega}(x) = \int_0^x \frac{\omega(t)}{t} dt + x \int_x^1 \frac{\omega(t)}{t^2} dt$$

is the conjugate modulus of continuity.

As shown in [17], Proposition 2 is, in a sense, sharp in the scale of Lipschitz spaces. Moreover, general Calderón–Zygmund operators map Λ^ω into $\Lambda^{\tilde{\omega}}$ even in the case, where the smoothness of $\partial\Omega$ does not affect the computations (see, for example, [9] for various results of this type). Also, we have $\tilde{\omega}_\alpha = \omega_\alpha$ for $\omega_\alpha(t) = t^\alpha$, $0 < \alpha < 1$. So, on the one hand, Proposition 2 is a sharp and direct extension of Theorem 1. On the other hand, Proposition 2 does not solve the B_Ω -invariance problem formulated in Section 1.1. In the present paper, we show that an appropriate choice for a B_Ω -invariant space $X(\Omega) \subset L^\infty(\Omega)$ is the Campanato space $\text{BMO}_\omega(\Omega)$ with a regular ω .

1.4. Restricted Beurling transform on Campanato spaces. In what follows, $Q \subset \mathbb{C}$ denotes a square with edges parallel to the coordinates axes, ℓ denotes the side length of Q , and $|Q| = \ell^2$ is the area of Q . Let dA denote the area measure on \mathbb{C} . Given $1 \leq p < \infty$ and a modulus of continuity ω , the Campanato space $\text{BMO}_\omega = \text{BMO}_{\omega,p}(\mathbb{C})$ consists of those $g \in L^p_{loc}(\mathbb{C})$ for which

$$(1.2) \quad \|g\|_{\text{BMO}_{\omega,p}} = \sup_{Q \subset \mathbb{C}} \frac{1}{\omega(\ell)} \|g - g_Q\|_{L^p(Q, dA/|Q|)} < \infty,$$

where

$$g_Q = \frac{1}{|Q|} \int_Q g(z) dA(z)$$

is the standard integral mean of g over Q . In fact, the arguments used in the studies of the classical space $\text{BMO}(\mathbb{R}^n)$, $n \geq 1$, guarantee that, for all $1 \leq p < \infty$, the seminorms under consideration define the same space (see, for example, [10], where methods from [3] are adapted). We use the equivalence of the corresponding seminorms in the proof of Theorem 3; see Section 3.2. So, below we consider the seminorm with $p = 1$ and we write BMO_ω in the place of $\text{BMO}_{\omega,1}$. The space BMO_ω is a generalization of Λ^α , $0 < \alpha < 1$: for $\omega_\alpha(t) = t^\alpha$, one has $\text{BMO}_{\omega_\alpha} = \Lambda^\alpha$ (see [2, 12]). In fact, $\text{BMO}_\omega \subset \Lambda^{\tilde{\omega}}$ for all Dini-regular ω ; see [15]. Also, the spaces $\text{BMO}_{\omega_\alpha} = \Lambda^\alpha$, $0 < \alpha < 1$, are known to be invariant for certain Calderón–Zygmund convolution operators (see [13]).

To define $\text{BMO}_\omega(\Omega)$ for a domain $\Omega \subset \mathbb{C}$, we use formula (1.2), replacing the norm in $L^p(Q, dA/|Q|)$ by that in $L^p(Q \cap \Omega, dA/|Q|)$ with $p = 1$. So, $\text{BMO}_\omega(\Omega)$ consists of those $f \in L^\infty(\Omega)$ for which

$$(1.3) \quad \|f\|_{\text{BMO}_\omega(\Omega)} = \sup_{Q \subset \mathbb{C}} \frac{1}{\omega(\ell)} \frac{1}{|Q|} \int_{Q \cap \Omega} |f(z) - f_{Q \cap \Omega}| dA(z) < \infty,$$

where

$$f_{Q \cap \Omega} = \frac{1}{|Q \cap \Omega|} \int_{Q \cap \Omega} f(z) dA(z)$$

is the integral mean of f over $Q \cap \Omega$.

Observe that in the case of the classical space $\text{BMO} = \text{BMO}(\mathbb{C})$, the above definition reduces to that of $\text{BMO}^r(\Omega)$, the restricted BMO on Ω (see, for example, [8]). While several BMO space are known for bounded domains, $\text{BMO}^r(\Omega)$ is usually considered as a standard one. For regular ω , we have $\text{BMO}_\omega \subset \Lambda^{\tilde{\omega}}$, hence, on the one hand, $g \in \text{BMO}_\omega$ implies that the restriction $g|_\Omega$ is in $L^\infty(\Omega)$ and $\|g|_\Omega\|_{\text{BMO}_\omega(\Omega)} < \infty$. On the other hand, if $f \in \text{BMO}_\omega(\Omega)$, then Lemma 8 provides an extension $\tilde{f} \in \text{BMO}_\omega$ such that $\tilde{f}|_\Omega = f$. Hence, $\text{BMO}_\omega(\Omega)$ is a weighted restricted BMO on Ω . Also, the property $\text{BMO}_\omega(\Omega) \subset L^\infty(\Omega)$ is a natural assumption in the definition of $\text{BMO}_\omega(\Omega)$.

The main result of the present paper guarantees that the space $\text{BMO}_\omega(\Omega)$ is B_Ω -invariant for all domains Ω and moduli of continuity ω under consideration.

Theorem 3. *Let ω be a regular modulus of continuity and let $\Omega \subset \mathbb{C}$ be a bounded domain with $\mathcal{C}^{1,\omega}$ -smooth boundary. Then the restricted Beurling transform B_Ω is bounded on $\text{BMO}_\omega(\Omega)$.*

Notation. The symbol C , with or without subscripts, denotes an absolute constant whose value may change from line to line. We write $E \lesssim F$ if $E \leq CF$ for a constant $C > 0$; also, $E \lesssim F \lesssim E$ is abbreviated as $E \approx F$.

A square on the complex plane \mathbb{C} is always a square with edges parallel to the coordinate axes. Given a square Q and a constant $a > 0$, aQ denotes the square with side length $a\ell(Q)$ and concentric with Q .

Organization of the paper. Auxiliary facts are collected in Section 2; in particular, we prove embedding and extension results (Lemmas 7 and 8, respectively). The main result, Theorem 3, is proved in Section 3.

2. AUXILIARY RESULTS

2.1. Regular moduli of continuity. Let ω be a regular modulus of continuity. First, property (1.1) trivially implies that the quotient $\omega(t)/t$ is almost decreasing, that is, there exists a constant $C > 0$ such that

$$\frac{\omega(t)}{t} \leq C \frac{\omega(s)}{s} \quad \text{for all } t > s > 0.$$

Secondly, (1.1) is known to be equivalent the following integral condition:

$$(2.1) \quad x \int_x^1 \frac{\omega(t)}{t^2} dt \leq C\omega(x)$$

for a positive constant C ; see, for example, [6, 7].

2.2. $\mathcal{C}^{1,\omega}$ -smooth boundary: technical properties and assumptions. Standard geometric arguments guarantee (see, for example, [11, 16] for related conditions in \mathbb{R}^n , $n \geq 2$) that $\Omega \subset \mathbb{C}$ is a bounded domain with a $\mathcal{C}^{1,\omega}$ -smooth boundary if and only if for each point $\zeta \in \partial\Omega$, there exists a square $Q = Q(\zeta)$ such that $\Omega \cap Q(\zeta)$ is a subgraph of a $\mathcal{C}^{1,\omega}$ function. More precisely, given $\zeta \in \partial\Omega$, we may assume (probably, after a suitable rotation by $\pm\frac{\pi}{2}$ or π) that the slope of the tangent to $\partial\Omega$ at ζ is positive. If the slope of the tangent is at least $1/10$, then consider a sufficiently small square $Q = Q(\zeta)$ such that ζ is the lower left vertex of Q . If $\partial\Omega$ intersects the right edge of Q , then let $I(\zeta)$ denote the lower edge of Q ; if $\partial\Omega$ intersects the upper edge of Q , then let $I(\zeta)$ denote the right edge of Q . Below we consider the first case; in the second case, we argue analogously after a suitable rotation. Also, if the slope of the tangent at ζ is less than $1/10$, then it suffices to consider a sufficiently small square Q such that ζ is the center of the left edge of Q and to define $I(\zeta)$ as the lower edge of Q .

So, assume that $\partial\Omega$ intersects the right edge of Q and $I(\zeta)$ is the lower edge of Q . Then there exists a function $\varphi \in \mathcal{C}^{1,\omega}$ such that the squares $Q(\zeta)$ and $Q'(\zeta)$ with edge $I(\zeta)$ have the following properties:

$$\begin{aligned} Q'(\zeta) &\subset \Omega, \\ \Omega \cap Q(\zeta) &= \{(x, y) = x + iy \in \mathbb{C} : (x, y_1) \in I(\zeta), y_1 \leq y < \varphi(x)\}, \\ \text{dist}(I'(\zeta), \partial\Omega) &\geq C\ell \quad \text{for a constant } C \in (0, 1), \end{aligned}$$

where $I'(\zeta) = \{(x, y_1 - \ell) : (x, y_1) \in I(\zeta)\}$ is the lower edge of $Q'(\zeta)$. Using compactness of $\partial\Omega$, we fix $r_0 > 0$ such that the above properties hold for all $\zeta \in \partial\Omega$ with $Q(\zeta)$ such that $\ell(Q(\zeta)) = r_0$. See Figure 1.

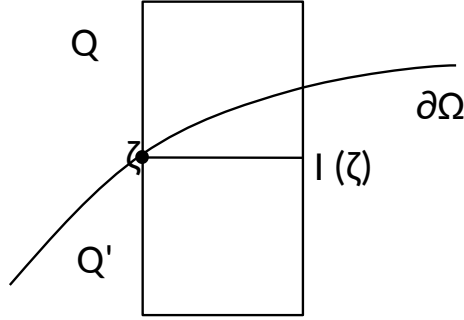


FIGURE 1.

2.3. Bloch spaces. Given a modulus of continuity ω , the Bloch space $\mathcal{B}_\omega(\Omega)$ consists of holomorphic in Ω functions f such that

$$\sup_{z \in \Omega} \frac{|f'(z)|\rho(z)}{\omega(\rho(z))} < \infty,$$

where $\rho(z)$ denotes the distance from z to $\partial\Omega$.

If the modulus of continuity ω is Dini-smooth and $\Omega \subset \mathbb{C}$ is a bounded domain with a $\mathcal{C}^{1,\omega}$ -smooth boundary, then standard arguments guarantee that

$$\mathcal{B}_\omega(\Omega) \subset L^\infty(\Omega) \subset L^1(\Omega).$$

The following result shows that the Beurling transform of the characteristic function χ_Ω is in an ω -weighted Bloch space.

Proposition 4 ([17, Theorem 2]). *Let ω be a regular modulus of continuity and let $\Omega \subset \mathbb{C}$ be a bounded domain with a $\mathcal{C}^{1,\omega}$ -smooth boundary. Then $B_\Omega \chi_\Omega \in \mathcal{B}_\omega(\mathbb{C} \setminus \partial\Omega)$.*

In fact, one may consider Proposition 4 as a Bloch-type analog of geometric characterizations of $\partial\Omega$ by Cruz and Tolsa [5] and Tolsa [16].

2.4. Integral means. An analog of the following fact is well-known for the classical space $\text{BMO}(\mathbb{C})$.

Lemma 5. *Let ω be a modulus of continuity and let $g \in \text{BMO}_\omega(\mathbb{C})$. Then*

$$|g_Q - g_{2Q}| \lesssim \omega(2\ell).$$

Proof. We have

$$\begin{aligned} |g_Q - g_{2Q}| &= \left| \frac{1}{|Q|} \int_Q g(z) dA(z) - g_{2Q} \right| \\ &\leq \frac{1}{|Q|} \int_Q |g(z) - g_{2Q}| dA(z) \\ &\leq \frac{1}{|Q|} \int_{2Q} |g(z) - g_{2Q}| dA(z) \\ &\leq 2\omega(2\ell) \end{aligned}$$

by the definitions of g_Q , g_{2Q} and BMO_ω . □

2.5. An embedding lemma. We will need the following basic observation (see also [6] for similar facts related to $\text{BMO}_\omega(\mathbb{R})$).

Lemma 6. *Let ω be a modulus of continuity and let $f \in L^\infty(\Omega)$. Assume that*

$$\sup_{Q \subset \mathbb{C}} \frac{1}{|Q|\omega(\ell)} \int_{Q \cap \Omega} |f(z) - b_Q| dA(z) = K < \infty$$

for some constants $b_Q \in \mathbb{C}$. Then $f \in \text{BMO}_\omega(\Omega)$.

Proof. For any square $Q \subset \mathbb{C}$, we have

$$\begin{aligned} \int_{Q \cap \Omega} |f(z) - f_{Q \cap \Omega}| dA(z) &\leq \int_{Q \cap \Omega} |f(z) - b_Q| dA(z) + \int_{Q \cap \Omega} |f_{Q \cap \Omega} - b_Q| dA(z) \\ &\leq K|Q|\omega(\ell) + \int_{Q \cap \Omega} \frac{1}{|Q \cap \Omega|} \int_{Q \cap \Omega} |f(w) - b_Q| dA(w) dA(z) \\ &\leq 2K|Q|\omega(\ell). \end{aligned}$$

So, $f \in \text{BMO}_\omega(\Omega)$. \square

Lemma 7. *Let ω be a regular modulus of continuity and let $\Omega \subset \mathbb{C}$ be a bounded domain with a $\mathcal{C}^{1,\omega}$ -smooth boundary. Then*

$$\mathcal{B}_\omega(\Omega) \subset \text{BMO}_\omega(\Omega).$$

Proof. First, as indicated in Section 2.3, $\mathcal{B}_\omega(\Omega) \subset L^\infty(\Omega)$. Next, to estimate the seminorm $\|Bf\|_{\text{BMO}_\omega(\Omega)}$, we have to consider the supremum over all squares Q , as defined by (1.3). Let $r_0 = r_0(\Omega) > 0$ be the constant fixed in Section 2.2. If $Q \subset \mathbb{C}$ is a square such that $\ell(Q) \geq r_0$, then

$$\frac{1}{|Q|} \int_Q |f(z)| dA(z) \lesssim \|f\|_{L^\infty(\Omega)} \lesssim \|f\|_{\mathcal{B}_\omega(\Omega)} \lesssim \omega(r_0).$$

Hence, it remains to consider the squares Q with $\ell(Q) < r_0$. If $Q \cap \Omega = \emptyset$, then the required estimate is trivial. So, let $Q \cap \Omega \neq \emptyset$. After a suitable rotation of \mathbb{C} (by $\pm \frac{\pi}{2}$ or π), we guarantee that the lower edge of Q intersects Ω . Moreover, we may assume that the lower edge of Q is in Ω , applying a suitable rotation and replacing Q by a smaller square such that $Q \cap \Omega$ is not changed, or shifting Q down so that the intersection $Q \cap \Omega$ becomes larger. If $Q \subset \Omega$, then let (b, y_1) denote the upper right vertex of Q else let (b, y_1) denote the intersection point of $\partial\Omega$ and the right edge of Q .

Using the properties of $Q \cap \Omega$ described in the definition of r_0 , we conclude that

$$Q \cap \Omega = \{(x, y) = x + iy \in \mathbb{C} : a \leq x \leq b, y_2 \leq y < \psi(x)\},$$

where $y_1 - \ell \leq y_2 \leq y_1$, $b - a = \ell < r_0$ and $\psi(x) = \min\{\varphi(x), y_2 + \ell\}$.

The function f is complex-valued; however, consideration of $\text{Re } f$ and $\text{Im } f$ allows to assume below in the proof that f is real-valued.

By Lemma 6, it suffices to prove that

$$(2.2) \quad \frac{1}{|Q|} \int_{Q \cap \Omega} |f - f_{y_0}| \lesssim \omega(\ell),$$

where $y_0 = y_1 - \ell$ and

$$f_{y_0} = \frac{1}{b-a} \int_a^b f(t, y_0) dt.$$

First, the function $f(t, y_0)$ is real-valued and continuous for $a \leq t \leq b$, thus, $f_{y_0} = f(s, y_0)$ for certain $s \in (a, b)$. Therefore, for any $x \in (a, b)$, we have

$$(2.3) \quad |f(x, y_0) - f_{y_0}| = |f(x, y_0) - f(s, y_0)| \leq \ell |\nabla f(\xi, y_0)|, \quad \text{where } \xi = \xi(x).$$

By the definition of $r_0 > 0$, we have $\text{dist}([a, b] \times y_0, \partial\Omega) \geq C\ell$ with $0 < C < 1$. Hence, applying (2.3), we obtain

$$(2.4) \quad |f(x, y_0) - f_{y_0}| \leq \ell \frac{\omega(C\ell)}{C\ell} \lesssim \omega(\ell) \quad \text{uniformly in } x \in (a, b),$$

since $\frac{\omega(t)}{t}$ is almost decreasing and $\omega(t)$ is increasing.

Secondly, we have

$$\begin{aligned} & \frac{1}{|Q|} \int_{Q \cap \Omega} |f(x, y) - f_{y_0}| dx dy \\ & \leq \frac{1}{|Q|} \int_{Q \cap \Omega} |f(x, y_0) - f_{y_0}| dx dy + \frac{1}{|Q|} \int_{Q \cap \Omega} |f(x, y) - f(x, y_0)| dx dy \\ & := E + F. \end{aligned}$$

Clearly, $E \lesssim \omega(\ell)$ by (2.4). Using the definition of $\mathcal{B}_\omega(\Omega)$ and Fubini's theorem, we obtain the following chain of estimates:

$$\begin{aligned} F & \lesssim \frac{1}{|Q|} \int_{Q \cap \Omega} \int_{y_0}^y |\nabla f(x, t)| dt dy dx \\ & \lesssim \frac{1}{|Q|} \int_a^b \int_{y_1}^{\psi(x)} \int_{y_0}^y \frac{\omega(\psi(x) - t)}{\psi(x) - t} dt dy dx \\ & = \frac{1}{|Q|} \int_a^b \int_{y_0}^{\psi(x)} \int_t^{\psi(x)} \frac{\omega(\psi(x) - t)}{\psi(x) - t} dy dt dx \\ & \lesssim \frac{1}{|Q|} \int_a^b \int_{y_0}^{\psi(x)} \omega(\psi(x) - t) dt dx \\ & \lesssim \frac{1}{\ell^2} \int_a^b 2\ell \omega(2\ell) dx \\ & \lesssim \omega(\ell), \end{aligned}$$

since $\omega(t)$ is increasing and $\frac{\omega(t)}{t}$ is almost decreasing. In sum, we obtain (2.2). So, the proof of the lemma is finished. \square

2.6. An extension lemma. The domain Ω has a sufficiently smooth boundary, hence, one could try to extend the argument from [8] to the ω -weighted case in the setting of appropriate domains in \mathbb{R}^n , $n \geq 2$. However, we apply a different argument, which uses the specifics of the complex plane.

The domain Ω has a $\mathcal{C}^{1,\omega}$ -smooth boundary, hence, there exists a bilipschitz map $\tau : \mathbb{C} \rightarrow \mathbb{C}$ such that $\tau(\infty) = \infty$ and $\tau(\mathbb{T}) = \partial\Omega$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$; see [14, Theorem 7.10]. Let τ_{-1} denote the inverse of τ . Then the map

$$(2.5) \quad \varkappa(z) = \tau \left(\frac{1}{\tau_{-1}} \right) (z), \quad z \in \mathbb{C},$$

is bilipschitz in a neighborhood of $\partial\Omega$. In the following lemma, using \varkappa , we extend each function from $\text{BMO}_\omega(\Omega)$ to an element of $\text{BMO}_\omega = \text{BMO}_\omega(\mathbb{C})$.

Lemma 8. *Let $\Omega \subset \mathbb{C}$ be a bounded domain with a $C^{1,\omega}$ -smooth boundary and let \varkappa be defined by (2.5). For $f \in \text{BMO}_\omega(\Omega)$, define*

$$\begin{aligned}\tilde{f}(z) &= f(z), \quad z \in \Omega, \\ \tilde{f}(z) &= f(\varkappa(z)), \quad z \in \mathbb{C} \setminus \Omega.\end{aligned}$$

Then $\tilde{f} \in \text{BMO}_\omega \cap L^\infty(\mathbb{C})$ and $\|\tilde{f}\|_{\text{BMO}_\omega} \lesssim \|f\|_{\text{BMO}_\omega(\Omega)}$.

Proof. We have $f \in \text{BMO}_\omega(\Omega) \subset L^\infty(\Omega)$, thus $\tilde{f} \in L^\infty(\mathbb{C})$. So, below we estimate the supremum defined by (1.2).

The maps τ and τ_{-1} are bilipschitz on \mathbb{C} , so all distances are distorted at most in M times, where M is bilipschitz constant of τ . Therefore, it suffices to consider the case, where Ω is the unit disk \mathbb{D} and $\varkappa(z) = \varkappa^{-1}(z) = 1/\bar{z}$.

Fix a constant $r_0 \in (0, 1/4)$. Let $Q \subset \mathbb{C}$ be a square. If $\ell(Q) \geq r_0$, then

$$\frac{1}{|Q|} \int_Q |f(z)| dA(z) \lesssim \|f\|_\infty \lesssim \omega(r_0).$$

So, it remains to consider the case, where $\ell(Q) \leq r_0$.

Q is in the complement of \mathbb{D} . Let $d = |w_0| > 1$, where w_0 denotes the center of Q . We have $\ell(Q) \leq r_0 < 1/4$ and

$$\frac{1}{d+2\ell} < \frac{1}{|z|} < \frac{1}{d-2\ell}, \quad z \in Q.$$

Hence, using elementary geometric arguments, we obtain a square \tilde{Q} such that $\varkappa(Q) \subset \tilde{Q} \subset \mathbb{D}$ and $\ell(\tilde{Q}) \approx \frac{\ell}{d^2}$.

So, we have the following chain of inequalities:

$$\begin{aligned}\frac{1}{|Q|} \int_Q |\tilde{f}(z) - f_{\tilde{Q}}| dA(z) &= \frac{1}{|Q|} \int_{\varkappa(Q)} |f(z) - f_{\tilde{Q}}| \frac{1}{|z|^4} dA(z) \\ &\lesssim \frac{d^4}{|Q|} \int_{\tilde{Q}} |f(z) - f_{\tilde{Q}}| dA(z) \\ &\lesssim \frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |f(z) - f_{\tilde{Q}}| dA(z) \\ &\lesssim \omega(\ell/d^2) \\ &\leq \omega(\ell).\end{aligned}$$

Q intersects \mathbb{D} . For $z \in Q$, we have

$$\frac{1}{1+2\ell} < \frac{1}{|z|} < \frac{1}{1-2\ell}.$$

Put $Q_0 = Q \cap \mathbb{D}$ and $Q_1 = Q \cap (\mathbb{C} \setminus \mathbb{D})$. Observe that the map \varkappa is bilipschitz in a sufficiently large neighborhood of $\partial\mathbb{D}$. Hence, applying standard geometric arguments, we obtain a square \tilde{Q} such that $\tilde{Q} \supset Q_0 \cup \varkappa(Q_1)$ and $|\tilde{Q}| \approx |Q|$. Since $Q_0 \sqcup Q_1 = Q$, we have

$$\begin{aligned}\frac{1}{|Q|} \int_Q |\tilde{f}(z) - f_{\tilde{Q}}| dA(z) &= \frac{1}{|Q|} \int_{Q_0} |f(z) - f_{\tilde{Q}}| dA(z) + \int_{Q_1} |\tilde{f}(z) - f_{\tilde{Q}}| dA(z) \\ &:= E + F.\end{aligned}$$

Clearly, $E \lesssim \omega(\ell)$. Since \varkappa is bilipschitz on Q_1 with a universal distortion constant, we also obtain the following chain of estimates:

$$\begin{aligned} F &= \frac{1}{|Q|} \int_{\varkappa(Q_1)} |f(z) - f_{\tilde{Q}}| \frac{1}{|z|^4} dA(z) \\ &\lesssim \frac{1}{|Q|} \int_{\tilde{Q} \cap \mathbb{D}} |f(z) - f_{\tilde{Q}}| dA(z) \\ &\lesssim \omega(\ell). \end{aligned}$$

Finally, applying Lemma 6 with $\Omega = \mathbb{C}$ and $b_Q = 0$ or $b_Q = f_{\tilde{Q}}$, we obtain $f \in \text{BMO}_\omega(\mathbb{C})$, as required. \square

3. PROOF OF THEOREM 3

Let $f \in \text{BMO}_\omega(\Omega)$. To estimate the seminorm $\|B_\Omega f\|_{\text{BMO}_\omega(\Omega)}$, we have to consider the supremum over all squares Q , as defined by (1.3). So, fix a square $Q \subset \Omega$. Let $\tilde{f} \in \text{BMO}_\omega$ be an extension of f provided by Lemma 8. Put

$$\begin{aligned} f_1 &= \tilde{f}_Q \chi_\Omega; \\ f_2 &= (f - \tilde{f}_Q) \chi_{2Q \cap \Omega}; \\ f_3 &= (f - \tilde{f}_Q) \chi_{\Omega \setminus 2Q}. \end{aligned}$$

Observe that $f = f_1 + f_2 + f_3$. So, to prove the theorem, it suffices to show that

$$\|B_\Omega f_j\|_{\text{MO}_{Q|\Omega}} := \frac{1}{|Q|} \int_{Q \cap \Omega} |B_\Omega f_j - (B_\Omega f_j)_{Q|\Omega}| \lesssim \omega(\ell), \quad j = 1, 2, 3.$$

3.1. First term. By Lemma 8, the extension \tilde{f} is bounded on \mathbb{C} , hence $|\tilde{f}_Q| \lesssim C$. Therefore,

$$(3.1) \quad \|B_\Omega f_1\|_{\text{MO}_{Q|\Omega}} = \|\tilde{f}_Q B_\Omega \chi_\Omega\|_{\text{MO}_{Q|\Omega}} \lesssim \|B_\Omega \chi_\Omega\|_{\text{MO}_{Q|\Omega}}.$$

Now, by Proposition 4 and Lemma 7, we have $B_\Omega \chi_\Omega \in \mathcal{B}_\omega(\Omega) \subset \text{BMO}_\omega(\Omega)$, that is, $\|B_\Omega \chi_\Omega\|_{\text{BMO}_\omega(\Omega)} \lesssim C = C(\Omega)$. So, combining with (3.1), we obtain

$$\|B_\Omega f_1\|_{\text{MO}_{Q|\Omega}} \lesssim \|B_\Omega \chi_\Omega\|_{\text{MO}_{Q|\Omega}} \leq \omega(\ell) \|B_\Omega \chi_\Omega\|_{\text{BMO}_\omega(\Omega)} \lesssim \omega(\ell),$$

as required.

3.2. Second term. We clearly have

$$\|B_\Omega f_2\|_{\text{MO}_{Q|\Omega}} \lesssim \frac{1}{|Q|} \int_{Q \cap \Omega} |B_\Omega f_2(z)| dA(z).$$

Next, applying the definition of f_2 , Hölder's inequality, boundedness of the Beurling transform on L^2 , a trivial integral inequality, and the triangle inequality, we obtain

the following chain of estimates:

$$\begin{aligned}
\frac{1}{|Q|} \int_{Q \cap \Omega} |B_\Omega f_2(z)| dA(z) &= \frac{1}{|Q|} \int_{Q \cap \Omega} |B_\Omega((f - \tilde{f}_Q)\chi_{2Q \cap \Omega})(z)| dA(z) \\
&\leq \left(\frac{1}{|Q|} \int_{Q \cap \Omega} |B_\Omega((f - \tilde{f}_Q)\chi_{2Q \cap \Omega})(z)|^2 dA(z) \right)^{\frac{1}{2}} \\
&\lesssim \left(\frac{1}{|2Q|} \int_{2Q \cap \Omega} |f(z) - \tilde{f}_Q|^2 dA(z) \right)^{\frac{1}{2}} \\
&\leq \left(\frac{1}{|2Q|} \int_{2Q} |\tilde{f}(z) - \tilde{f}_Q|^2 dA(z) \right)^{\frac{1}{2}} \\
&\leq |\tilde{f}_{2Q} - \tilde{f}_Q| + \left(\frac{1}{|2Q|} \int_{2Q} |\tilde{f}(z) - \tilde{f}_{2Q}|^2 dA(z) \right)^{\frac{1}{2}} \\
&:= E + F.
\end{aligned}$$

First, $E \lesssim \omega(\ell)$ by Lemma 5. Secondly,

$$F \leq \omega(\ell) \|\tilde{f}\|_{\text{BMO}_{\omega,2}}.$$

As mentioned in the introduction, the seminorm in $\text{BMO}_{\omega,2}$ is equivalent to that in $\text{BMO}_{\omega,1} = \text{BMO}_\omega$. Thus $F \lesssim \omega(\ell)$.

3.3. Third term. Put $\tilde{f}_3 = \tilde{f} - \tilde{f}_Q$. By the definitions of B_Ω and f_3 , we have

$$\begin{aligned}
&\pi|Q| \|B_\Omega f_3\|_{\text{MO}_{Q|\Omega}} \\
&= \int_{Q \cap \Omega} \left| \int_{\Omega \setminus 2Q} \frac{\tilde{f}_3(u) dA(u)}{(u-z)^2} - \frac{1}{|Q \cap \Omega|} \int_{Q \cap \Omega} \int_{\Omega \setminus 2Q} \frac{\tilde{f}_3(u) dA(u)}{(u-w)^2} dA(w) \right| dA(z) \\
&= \int_{Q \cap \Omega} \frac{1}{|Q \cap \Omega|} \int_{Q \cap \Omega} \int_{\Omega \setminus 2Q} |\tilde{f}_3(u)| \left| \frac{1}{(u-z)^2} - \frac{1}{(u-w)^2} \right| dA(u) dA(w) dA(z).
\end{aligned}$$

Fix a point $z_0 \in Q \cap \Omega$. For $z, w \in Q \cap \Omega$ and $u \in \Omega \setminus 2Q$, we have

$$\left| \frac{1}{(u-z)^2} - \frac{1}{(u-w)^2} \right| \lesssim \frac{|z-w|}{|u-z|^3} \lesssim \frac{\ell}{|u-z_0|^3}.$$

Therefore,

$$\begin{aligned}
\|B_\Omega f_3\|_{\text{MO}_{Q|\Omega}} &\lesssim \frac{1}{|Q|} \int_{Q \cap \Omega} \frac{1}{|Q \cap \Omega|} \int_{Q \cap \Omega} \int_{\Omega \setminus 2Q} |\tilde{f}_3(u)| \frac{|z-w|}{|u-z|^3} dA(u) dA(w) dA(z) \\
&\lesssim \ell \int_{\Omega \setminus 2Q} \frac{|\tilde{f}(u) - \tilde{f}_Q|}{|u-z_0|^3} dA(u).
\end{aligned}$$

Put $Q_k = \Omega \cap (2^{k+1}Q \setminus 2^kQ)$, $k = 1, 2, \dots$. So

$$\Omega \setminus 2Q = \bigcup_{k=1}^{\infty} Q_k.$$

For $u \in Q_k$, we have and $|u - z_0| \approx 2^k \ell$. Hence,

$$\begin{aligned} \|B_\Omega f_3\|_{\text{MO}_{Q|\Omega}} &\lesssim \sum_{k=1}^{\infty} \frac{\ell}{(2^k \ell)^3} \int_{Q_k} |\tilde{f}(u) - \tilde{f}_Q| dA(u) \\ &= \sum_{1 \leq k < m} + \sum_{k=m}^{\infty}, \end{aligned}$$

where m is the smallest integer such that $2^m \ell \geq \frac{1}{2}$. Also, the first sum is equal to zero if $m = 1$.

First, $|\tilde{f} - \tilde{f}_Q| \lesssim C$ and $|Q_k| \lesssim (2^k \ell)^2$, thus

$$\sum_{k=m}^{\infty} \lesssim \sum_{k=m}^{\infty} \frac{\ell |Q_k|}{(2^k \ell)^3} \lesssim 2^{-m} \lesssim \omega(\ell).$$

Secondly, using a telescoping sum, definition (1.2) and Lemma 5, we obtain the following chain of estimates:

$$\begin{aligned} &\int_{Q_k} |\tilde{f}(u) - \tilde{f}_Q| dA(u) \\ &\lesssim \int_{Q_k} |\tilde{f}(u) - \tilde{f}_{2^k Q}| dA(u) + \int_{2^k Q} \sum_{n=0}^{k-1} |\tilde{f}_{2^{n+1} Q} - \tilde{f}_{2^n Q}| dA(u) \\ &\lesssim (2^k \ell)^2 \sum_{n=1}^k \omega(2^n \ell). \end{aligned}$$

Therefore,

$$\sum_{k=1}^{m-1} \lesssim \ell \sum_{k=1}^{m-1} \frac{(2^k \ell)^2}{(2^k \ell)^3} \sum_{n=1}^k \omega(2^n \ell)$$

So, changing the summation order, we obtain

$$\begin{aligned} \sum_{k=1}^{m-1} &\lesssim \ell \sum_{n=1}^{m-1} \omega(2^n \ell) \sum_{k=n}^{m-1} \frac{1}{2^k \ell} \\ &\lesssim \ell \sum_{n=1}^{m-1} \frac{\omega(2^n \ell)}{2^n \ell} \\ &\lesssim \ell \int_{2\ell}^1 \frac{\omega(t)}{t^2} dt \\ &\lesssim \omega(\ell), \end{aligned}$$

by (2.1). In sum, we have

$$\|B_\Omega f_3\|_{\text{MO}_{Q|\Omega}} \lesssim \omega(\ell).$$

So, the proof of the theorem is finished.

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